Finitely-additive measures on the asymptotic foliations of a Markov compactum.

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1 Introduction.

1.1 Hölder cocycles over translation flows.

Let $\rho \geq 2$ be an integer, let M be a compact orientable surface of genus ρ, and let ω be a holomorphic one-form on M. Denote by $\mathfrak{m} = (\omega \wedge \overline{\omega})/2i$ the area form induced by ω and assume that $\mathfrak{m}(M) = 1$.

Let h_t^+ be the *vertical* flow on M (i.e., the flow corresponding to $\Re(\omega)$); let h_t^- be the *horizontal* flow on M (i.e., the flow corresponding to $\Im(\omega)$). The flows h_t^+, h_t^- preserve the area $\mathfrak{m}$ and are uniquely ergodic.

Take $x \in M$, $t_1, t_2 \in \mathbb{R}_+$ and assume that the closure of the set

$$\{h^+_{\tau_1} h^-_{\tau_2} x, 0 \leq \tau_1 < t_1, 0 \leq \tau_2 < t_2\} \tag{1}$$

does not contain zeros of the form ω. Then the set (1) is called *an admissible rectangle* and denoted $\Pi(x, t_1, t_2)$. Let $\overline{\mathfrak{C}}$ be the semi-ring of admissible rectangles.

Consider the linear space $\mathcal{Y}^+$ of Hölder cocyles $\Phi^+(x, t)$ over the vertical flow h_t^+ which are invariant under horizontal holonomy. More precisely, a function $\Phi^+(x, t) : M \times \mathbb{R} \to \mathbb{C}$ belongs to the space $\mathcal{Y}^+$ if it satisfies:

1. $\Phi^+(x, t + s) = \Phi^+(x, t) + \Phi^+(h_t^+ x, s)$;
2. There exists $t_0 > 0$, $\theta > 0$ such that $|\Phi^+(x, t)| \leq t^\theta$ for all $x \in M$ and all $t \in \mathbb{R}$ satisfying $|t| < t_0$;
3. If $\Pi(x, t_1, t_2)$ is an admissible rectangle, then $\Phi^+(x, t_1) = \Phi^+(h^-_{t_2} x, t_1)$.

For example, if a cocycle Φ_1^+ is defined by $\Phi_1^+(x, t) = t$, then clearly $\Phi_1^+ \in \mathcal{Y}^+$.

In the same way define the space of $\mathcal{Y}^-$ of Hölder cocyles $\Phi^-(x, t)$ over the horizontal flow h_t^- which are invariant under vertical holonomy, and set $\Phi_1^-(x, t) = t$.

Given $\Phi^+ \in \mathcal{Y}^+$, $\Phi^- \in \mathcal{Y}^-$, a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\overline{\mathfrak{C}}$ of admissible rectangles is introduced by the formula

$$\Phi^+ \times \Phi^-(\Pi(x, t_1, t_2)) = \Phi^+(x, t_1) \cdot \Phi^-(x, t_2). \tag{2}$$

In particular, for $\Phi^- \in \mathcal{Y}^-$, set $m_{\Phi^-} = \Phi_1^+ \times \Phi^-$:

$$m_{\Phi^-}(\Pi(x, t_1, t_2)) = t_1 \Phi^-(x, t_2). \tag{3}$$

For any $\Phi^- \in \mathcal{Y}^-$ the measure m_{Φ^-} satisfies $(h_t^+)_* m_{\Phi^-} = m_{\Phi^-}$ and is an invariant distribution in the sense of G. Forni [5], [6]. For instance, $m_{\Phi_1^-} = \mathfrak{m}$.

A $\mathbb{C}$-linear pairing between $\mathcal{Y}^+$ and $\mathcal{Y}^-$ is given, for $\Phi^+ \in \mathcal{Y}^+$, $\Phi^- \in \mathcal{Y}^-$, by the formula

$$< \Phi^+, \Phi^- >= \Phi^+ \times \Phi^-(M) \tag{4}$$

The space of Lipschitz functions is not invariant under h_t^+, and a larger function space $Lip_w^+(M, \omega)$ of weakly Lipschitz functions is introduced as follows. A bounded measurable function f belongs to $Lip_w^+(M, \omega)$ if there exists a constant C, depending only on f, such that for any admissible rectangle $\Pi(x, t_1, t_2)$ we have

$$\left| \int_0^{t_1} f(h_t^+ x) dt - \int_0^{t_1} f(h_t^+ (h_{t_2}^- x) dt \right| \le C. \tag{5}$$

Let C_f be the infimum of all C satisfying (5). We norm $Lip_w^+(X)$ by setting

$$||f||_{Lip_w^+} = \sup_X f + C_f.$$

By definition, the space $Lip_w^+(M, \omega)$ contains all Lipschitz functions on M and is invariant under h_t^+. We denote by $Lip_{w,0}^+(M, \omega)$ the subspace of $Lip_w^+(M, \omega)$ of functions whose integral with respect to $\mathfrak{m}$ is 0.

1.2 Flows along the stable foliation of a pseudo-Anosov diffeomorphism.

Assume that $\theta_1 > 0$ and a diffeomorphism $g : M \to M$ are such that

$$g^*(\Re(\omega)) = \exp(\theta_1)\Re(\omega);\ g^*(\Im(\omega)) = \exp(-\theta_1)\Im(\omega). \tag{6}$$

The diffeomorphism g induces a linear automorphism g^* of the cohomology space $H^1(M, \mathbb{C})$. Denote by E^+ the expanding subspace of g^* (in other words, E^+ is the subspace spanned by vectors corresponding to Jordan cells of g^* with eigenvalues exceeding 1 in absolute value). The action of g on $\mathcal{Y}^+$ is given by $g^*\Phi^+(x, t) = \Phi^+(gx, \exp(\theta_1)t)$.

Proposition 1 *There exists a g^*-equivariant isomorphism between E^+ and $\mathcal{Y}^+$.*

Theorem 1 *There exists a continuous mapping $\Xi^+ : Lip_w^+(M, \omega) \to \mathcal{Y}^+$ such that for any $f \in Lip_w^+(M, \omega)$, any $x \in X$ and any $T > 0$ we have*

$$\left| \int_0^T f \circ h_t^+(x) dt - \Xi^+(f)\big(x, T\big) \right| < C_\varepsilon ||f||_{Lip_w^+} (1 + \log(1 + T))^{2\rho+1}.$$

The mapping Ξ^+ satisfies $\Xi^+(f \circ h_t^+) = \Xi^+(f)$ and $\Xi^+(f \circ g) = g^*\Xi^+(f)$.

The mapping Ξ^+ is constructed as follows. By Proposition 1 applied to the flow h_t^-, there exists a g-equivariant isomorphism between $\mathcal{Y}^-$ and the contracting space for the action of g^* on $H^1(M, \mathbb{C})$ (in other words, the subspace spanned by vectors corresponding to Jordan cells with eigenvalues strictly less than 1 in absolute value).

Proposition 2 *The pairing* $<,>$ *given by (4) is nondegenerate and* g^**-invariant.*

Remark. Under the identification of $\mathcal{Y}^+$ and $\mathcal{Y}^-$ with respective subspaces of $H^1(M, \mathbb{C})$, the pairing $<,>$ is taken to the cup-product on $H^1(M, \mathbb{C})$ (see Proposition 4.19 in Veech [14]).

If $f \in Lip_w^+(M, \omega)$, then f is Riemann-integrable with respect to m_{Φ^-} for any $\Phi^- \in \mathcal{Y}^-$ (see (30) for a precise definition of the integral). Assign to f a cocycle Φ_f^+ in such a way that for all $\Phi^- \in \mathcal{Y}^-$ we have

$$< \Phi_f^+, \Phi^- >= \int_M f dm_{\Phi^-}. \tag{7}$$

By definition, $\Phi^+_{f \circ h_t^+} = \Phi_f^+$. The mapping Ξ^+ of Theorem 1 is given by the formula

$$\Xi^+(f) = \Phi_f^+. \tag{8}$$

The first eigenvalue for the action of g^* on E^+ is $\exp(\theta_1)$ and is always simple. If its second eigenvalue has the form $\exp(\theta_2)$, where $\theta_2 > 0$, and is simple as well, then the following limit theorem holds for h_t^+.

Given a bounded measurable function $f : X \to \mathbb{R}$ and $x \in X$, introduce a continuous function $\mathfrak{S}_n[f, x]$ on the unit interval by the formula

$$\mathfrak{S}_n[f, x](\tau) = \int_0^{\tau \exp(n\theta_1)} f \circ h_t^+(x) dt. \tag{9}$$

The functions $\mathfrak{S}_n[f, x]$ are $C[0, 1]$-valued random variables on the probability space $(M, \mathfrak{m})$.

Theorem 2 *If* $g^*|_{E^+}$ *has a simple, real second eigenvalue* $\exp(\theta_2)$*,* $\theta_2 > 0$*, then there exists a continuous functional* $\alpha : Lip_w^+(M, \omega) \to \mathbb{R}$ *and a compactly supported non-degenerate measure* η *on* $C[0, 1]$ *such that for any* $f \in Lip_{w,0}^+(M, \omega)$ *satisfying* $\alpha(f) \neq 0$ *the sequence of random variables*

$$\frac{\mathfrak{S}_n[f, x]}{\alpha(f) \exp(n\theta_2)}$$

converges in distribution to η *as* $n \to \infty$*.*

The functional α is constructed explicitly as follows. Under the assumptions of the theorem the action of g^* on E^- has a simple eigenvalue $\exp(-\theta_2)$; let $v(2)$ be the eigenvector with eigenvalue $\exp(-\theta_2)$, let $\Phi_2^- \in \mathcal{Y}^-$ correspond to $v(2)$ by Proposition 1 and $m_{\Phi_2^-}$ be given by (3); then

$$\alpha(f) = \int f dm_{\Phi_2^-}.$$

1.3 Generic translation flows.

Let $\rho \geq 2$ and let $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$ be a nonnegative integer vector such that $\kappa_1 + \cdots + \kappa_\sigma = 2\rho - 2$. Denote by $\mathcal{M}_\kappa$ the moduli space of Riemann surfaces of genus ρ endowed with a holomorphic differential of area 1 with singularities of orders $k_1, \ldots, k_\sigma$ (the *stratum* in the moduli space of holomorphic differentials), and let $\mathcal{H}$ be a connected component of $\mathcal{M}_\kappa$. Denote by g_t the Teichmüller flow on $\mathcal{H}$ (see [6], [8]), and let $\mathbb{A}(t, X)$ be the Kontsevich-Zorich cocycle over g_t [8].

Let $\mathbb{P}$ be a g_t-invariant ergodic probability measure on $\mathcal{H}$. For $X \in \mathcal{H}$, $X = (M, \omega)$, let $\mathcal{Y}_X^+$, $\mathcal{Y}_X^-$ be the corresponding spaces of Hölder cocycles. Denote by E_X^+ the space spanned by the positive Lyapunov exponents of the Kontsevich-Zorich cocycle.

Proposition 3 *For $\mathbb{P}$-almost all $X \in \mathcal{H}$, we have $\dim \mathcal{Y}_X^+ = \dim \mathcal{Y}_X^- = \dim E_X^+$, and the pairing $<,>$ between $\mathcal{Y}_X^+$ and $\mathcal{Y}_X^-$ is non-degenerate.*

Remark. In particular, if $\mathbb{P}$ is the Masur-Veech "smooth" measure [10, 12], then $\dim \mathcal{Y}_X^+ = \dim \mathcal{Y}_X^- = \rho$.

Assign to $f \in Lip_w^+(M, \omega)$ a cocycle Φ_f^+ by (7).

Theorem 3 *For any $\varepsilon > 0$ there exists a constant C_ε depending only on $\mathbb{P}$ such that for $\mathbb{P}$-almost every $X \in \mathcal{H}$, any $f \in Lip_w^+(X)$, any $x \in X$ and any $T > 0$ we have*

$$\Big| \int_0^T f \circ h_t^+(x) dt - \Phi_f^+(x, T) \Big| < C_\varepsilon ||f||_{Lip_w^+}(1 + T^\varepsilon).$$

If both the first and the second Lyapunov exponent of the measure $\mathbb{P}$ are positive and simple (as, by the Avila-Viana Theorem [2], is the case with the Masur-Veech "smooth" measure on $\mathcal{H}$), then the following limit theorem holds.

As before, consider a $C[0,1]$-valued random variable $\mathfrak{S}_t[f, x]$ on $(M, \mathfrak{m})$ defined by the formula

$$\mathfrak{S}_s[f, x](\tau) = \int_0^{\tau \exp(s)} f \circ h_t^+(x) dt.$$

Let $||v||$ be the Hodge norm in $H^1(M, \mathbb{R})$. Let $\theta_2 > 0$ be the second Lyapunov exponent of the Kontsevich-Zorich cocycle and let $v_2(X)$ be a Lyapunov vector corresponding to θ_2 (by our assumption, such a vector is unique up to scalar multiplication). Introduce a real-valued multiplicative cocycle $H_2(t, X)$ over g_t by the formula

$$H_2(t, X) = \frac{||A(t, X) v_2(X)||}{||v_2(X)||}. \tag{10}$$

Theorem 4 *Assume that both the first and the second Lyapunov exponent of the Kontsevich-Zorich cocycle with respect to the measure $\mathbb{P}$ are positive and simple. Then for $\mathbb{P}$-almost any $X' \in \mathcal{H}$ there exists a non-degenerate compactly supported measure $\eta_{X'}$ on $C[0,1]$ and, for $\mathbb{P}$-almost all $X, X' \in \mathcal{H}$, there exists a*

sequence of moments $s_n = s_n(X, X')$ such that the following holds. For $\mathbb{P}$-almost every $X \in \mathcal{H}$ there exists a continuous functional

$$\mathfrak{a}^{(X)} : Lip_w^+(X) \to \mathbb{R}$$

such that for $\mathbb{P}$-almost every X' and for any real-valued $f \in Lip_{w,0}^+(X)$ satisfying $\mathfrak{a}^{(X)}(f) \neq 0$, the sequence of $C[0,1]$-valued random variables

$$\frac{\mathfrak{S}_{s_n}[f,x](\tau)}{\left(\mathfrak{a}^{(X)}(f)\right) H_2(s_n, X)}$$

converges in distribution to $\eta_{X'}$ as $n \to \infty$.

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2 Asymptotic foliations of a Markov compactum.

2.1 Definitions and notation.

Let $m \in \mathbb{N}$ and let Γ be an oriented graph with m vertices $\{1, \ldots, m\}$ and possibly multiple edges. We assume that that for each vertex there is an edge starting from it and an edge ending in it.

Let $\mathcal{E}(\Gamma)$ be the set of edges of Γ. For $e \in \mathcal{E}(\Gamma)$ we denote by $I(e)$ its initial vertex and by $F(e)$ its terminal vertex. Let Q be the incidence matrix of Γ defined by the formula

$$Q_{ij} = \#\{e \in \mathcal{E}(\Gamma) : I(e) = i, F(e) = j\}.$$

By assumption, all entries of the matrix Q are positive. A finite word $e_1 \ldots e_k$, $e_i \in \mathcal{E}(\Gamma)$, will be called *admissible* if $F(e_{i+1}) = I(e_i)$, $i = 1, \ldots, k$.

To the graph Γ we assign a *Markov compactum* X_Γ, the space of bi-infinite paths along the edges:

$$X_\Gamma = \{x = \ldots x_{-n} \ldots x_0 \ldots x_n \ldots,\ x_n \in \mathcal{E}(\Gamma), F(x_{n+1}) = I(x_n)\}.$$

Remark. As Γ will be fixed throughout this section, we shall often omit the subscript Γ from notation and only insert it when the dependence on Γ is underlined.

Cylinders in X_Γ are subsets of the form $\{x : x_{n+1} = e_1, \ldots, x_{n+k} = e_k\}$, where $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $e_1 \ldots e_k$ is an admissible word. The family of all cylinders forms a semi-ring which we denote by $\mathfrak{C}$.

For $x \in X$, $n \in \mathbb{Z}$, introduce the sets

$$\gamma_n^+(x) = \{x' \in X_\Gamma : x'_t = x_t, t \geq n\};\ \gamma_n^-(x) = \{x' \in X_\Gamma : x'_t = x_t, t \leq n\};$$

$$\gamma_\infty^+(x) = \bigcup_{n\in\mathbb{Z}} \gamma_n^+(x);\ \gamma_\infty^-(x) = \bigcup_{n\in\mathbb{Z}} \gamma_n^-(x).$$

The sets $\gamma_\infty^+(x)$ are leaves of the asymptotic foliation $\mathcal{F}^+$ on the space X_Γ; the sets $\gamma_\infty^+(x)$ are leaves of the asymptotic foliation $\mathcal{F}^-$ on X_Γ.

For $n \in \mathbb{Z}$ let $\mathfrak{C}_n^+$ be the collection of all subsets of X_Γ of the form $\gamma_n^+(x)$, $n \in \mathbb{Z}$, $x \in X$; similarly, $\mathfrak{C}_n^-$ is the collection of all subsets of the form $\gamma_n^-(x)$. Set

$$\mathfrak{C}^+ = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}_n^+;\ \mathfrak{C}^- = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}_n^-. \tag{11}$$

The collection $\mathfrak{C}_n^+$ is a semi-ring for any $n \in \mathbb{Z}$. Since every element of $\mathfrak{C}_n^+$ is a disjoint union of elements of $\mathfrak{C}_{n+1}^+$, the collection $\mathfrak{C}^+$ is a semi-ring as well. The same statements hold for $\mathfrak{C}_n^-$ and $\mathfrak{C}^-$.

Let $\exp(\theta_1)$ be the spectral radius of the matrix Q, and let $h = (h_1, \ldots, h_m)$ be the unique positive eigenvector of Q: we thus have $Qh = \exp(\theta_1)h$. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be the positive eigenvector of the transpose matrix Q^t: we have $Q^t\lambda = \exp(\theta_1)\lambda$. The vectors λ, h are normalized as follows:

$$\sum_{i=1}^m \lambda_i = 1;\ \sum_{i=1}^m \lambda_i h_i = 1. \tag{12}$$

Introduce a sigma-additive positive measure Φ_1^+ on the semi-ring $\mathfrak{C}^+$ by the formula

$$\Phi_1^+(\gamma_n^+(x)) = h_{F(x_n)} \exp((n-1)\theta_1) \tag{13}$$

and a sigma-additive positive measure Φ_1^- on the semi-ring $\mathfrak{C}^-$ by the formula

$$\Phi_1^-(\gamma_n^-(x)) = \lambda_{I(x_n)} \exp(-n\theta_1). \tag{14}$$

Let $n \in \mathbb{Z}$, $k \in \mathbb{N}$, and let $e_1 \ldots e_k$ be an admissible word. The Parry measure ν on X_Γ is defined by the formula

$$\nu(\{x : x_{n+1} = e_1, \ldots, x_{n+k} = e_k\}) = \lambda_{I(e_k)} h_{F(e_1)} \exp(-k\theta_1). \tag{15}$$

The measures Φ_1^+, Φ_1^- are conditional measures of the Parry measure ν in the following sense. If $C \in \mathfrak{C}$, then $\gamma_\infty^+(x) \cap C \in \mathfrak{C}^+$, $\gamma_\infty^-(x) \cap C \in \mathfrak{C}^-$ for any $x \in C$, and we have

$$\nu(C) = \Phi_1^+(\gamma_\infty^+(x) \cap C) \cdot \Phi_1^-(\gamma_\infty^-(x) \cap C). \tag{16}$$

2.2 Finitely-additive measures on leaves of asymptotic foliations.

Given $v \in \mathbb{C}^m$, write

$$|v| = \sum_{i=1}^{m} |v_i|. \tag{17}$$

The norms of all matrices in this paper are understood with respect to this norm. Consider the direct-sum decomposition

$$\mathbb{C}^m = E^+ \oplus E^-,$$

where E^+ is spanned by Jordan cells of eigenvalues of Q with absolute value exceeding 1, and E^- is spanned by Jordan cells corresponding to eigenvalues of Q with absolute value at most 1. Let $v \in E^+$ and for all $n \in \mathbb{Z}$ set $v^{(n)} = Q^n v$ (note that $Q|_{E^+}$ is by definition invertible). Introduce a finitely-additive complex-valued measure Φ_v^+ on the semi-ring $\mathfrak{C}^+$ (defined in (11)) by the formula

$$\Phi_v^+(\gamma_{n+1}^+(x)) = (v^{(n)})_{F(x_{n+1})}. \tag{18}$$

The measure Φ_v^+ is invariant under holonomy along $\mathcal{F}^-$: by definition, we have the following

Proposition 4 *If $F(x_n) = F(x'_n)$, then $\Phi_v^+(\gamma_n^+(x)) = \Phi_v^+(\gamma_n^+(x'))$.*

The measures Φ_v^+ span a complex linear space, which we denote $\mathcal{Y}^+$ (or, sometimes, $\mathcal{Y}_\Gamma^+$, when dependence on Γ is stressed.) The map

$$\mathcal{I} : v \to \Phi_v^+ \tag{19}$$

is an isomorphism between E^+ and $\mathcal{Y}_\Gamma^+$.

For Q^t, we have the direct-sum decomposition

$$\mathbb{C}^m = \tilde{E}^+ \oplus \tilde{E}^-,$$

where $\tilde{E}^+$ is spanned by Jordan cells of eigenvalues of Q^t with absolute value exceeding 1, and $\tilde{E}^-$ is spanned by Jordan cells corresponding to eigenvalues of Q^t with absolute value at most 1. As before, for $\tilde{v} \in \tilde{E}^+$ set $\tilde{v}^{(n)} = (Q^t)^n \tilde{v}$ for all $n \in \mathbb{Z}$, and introduce a finitely-additive complex-valued measure $\Phi_{\tilde{v}}^-$ on the semi-ring $\mathfrak{C}^-$ (defined in (11)) by the formula

$$\Phi_{\tilde{v}}^-(\gamma_n^-(x)) = (\tilde{v}^{(-n)})_{I(x_n)}. \tag{20}$$

By definition, the measure $\Phi_{\tilde{v}}^-$ is invariant under holonomy along $\mathcal{F}^+$: more precisely, we have the following

Proposition 5 *If $I(x_n) = I(x'_n)$, then $\Phi_{\tilde{v}}^-(\gamma_n^-(x)) = \Phi_{\tilde{v}}^-(\gamma_n^-(x'))$.*

Let $\mathcal{Y}^-_\Gamma$ be the space spanned by the measures Φ^-_v, $v \in \tilde{E}^+$. The map

$$\tilde{\mathcal{I}} : v \to \Phi^-_v \tag{21}$$

is an isomorphism between $\tilde{E}^+$ and $\mathcal{Y}^-_\Gamma$.

Let $\sigma : X_\Gamma \to X_\Gamma$ be the shift defined by $(\sigma x)_i = x_{i+1}$. The shift σ naturally acts on the spaces $\mathcal{Y}^+_\Gamma$, $\mathcal{Y}^-_\Gamma$: given $\Phi \in \mathcal{Y}^+_\Gamma$ (or $\mathcal{Y}^-_\Gamma$), the measure $\sigma_*\Phi$ is defined, for $\gamma \in \mathfrak{C}^+$, by the formula

$$\sigma_*\Phi(\gamma) = \Phi(\sigma\gamma).$$

From the definitions we obtain

Proposition 6 *The following diagrams are commutative:*

$$\begin{array}{ccc} E^+ & \xrightarrow{\mathcal{I}} & \mathcal{Y}^+_\Gamma \\ \Big\downarrow{\scriptstyle Q} & & \Big\uparrow{\scriptstyle \sigma^*} \\ E^+ & \xrightarrow{\mathcal{I}} & \mathcal{Y}^+_\Gamma \end{array}$$

$$\begin{array}{ccc} \tilde{E}^+ & \xrightarrow{\tilde{\mathcal{I}}} & \mathcal{Y}^-_\Gamma \\ \Big\downarrow{\scriptstyle Q^t} & & \Big\downarrow{\scriptstyle \sigma^*} \\ \tilde{E}^+ & \xrightarrow{\tilde{\mathcal{I}}} & \mathcal{Y}^-_\Gamma \end{array}$$

2.3 Pairings.

Given $\Phi^+ \in \mathcal{Y}^+$, $\Phi^- \in \mathcal{Y}^-$, introduce, in analogy with (16), a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\mathfrak{C}$ of cylinders in X_Γ: for any $C \in \mathfrak{C}$ and $x \in C$, set

$$\Phi^+ \times \Phi^-(C) = \Phi^+(\gamma^+_\infty(x) \cap C) \cdot \Phi^-(\gamma^-_\infty(x) \cap C). \tag{22}$$

Note that by Propositions 4, 5, the right-hand side in (22) does not depend on $x \in C$.

More explicitly, let $v \in E^+$, $\tilde{v} \in \tilde{E}^+$, $\Phi^+_v = \mathcal{I}(v)$, $\Phi^-_{\tilde{v}} = \tilde{\mathcal{I}}(\tilde{v})$. As above, denote $v^{(n)} = Q^n v$, $\tilde{v}^{(n)} = (Q^t)^n v$. Let $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and let $e_1 \dots e_k$ be an admissible word. Then

$$\Phi^+_v \times \Phi^-_{\tilde{v}}(\{x : x_{n+1} = e_1, \dots, x_{n+k} = e_k\}) = \big(v^{(n)}\big)_{F(e_1)}\big(\tilde{v}^{(-n-k)}\big)_{I(e_{n+k})}. \tag{23}$$

There is a natural $\mathbb{C}$-linear pairing $<,>$ between the spaces $\mathcal{Y}^+_\Gamma$ and $\mathcal{Y}^-_\Gamma$: for $\Phi^+ \in \mathcal{Y}^+_\Gamma$, $\Phi^- \in \mathcal{Y}^-_\Gamma$, set

$$< \Phi^+, \Phi^- >= \Phi^+ \times \Phi^-(X_\Gamma). \tag{24}$$

From (23) we derive

Proposition 7 *Let $v \in E^+$, $\tilde{v} \in \tilde{E}^+$, $\Phi_v^+ = \mathcal{I}_\Gamma(v)$, $\Phi_{\tilde{v}}^- = \tilde{\mathcal{I}}_\Gamma(\tilde{v})$. Then*

$$< \Phi_v^+, \Phi_{\tilde{v}}^- >= \sum_{i=1}^m v_i \tilde{v}_i. \tag{25}$$

In particular, the pairing $<,>$ is non-degenerate and σ^*-invariant.

In particular, for $\Phi^- \in \mathcal{Y}^-$ denote

$$m_{\Phi^-} = \Phi_1^+ \times \Phi^-. \tag{26}$$

2.4 Weakly Lipschitz Functions.

Introduce a function space $Lip_w^+(X)$ in the following way. A bounded Borel-measurable function $f : X \to \mathbb{C}$ belongs to the space $Lip_w^+(X)$ if there exists a constant $C > 0$ such that for all $n \geq 0$ and any $x, x' \in X$ satisfying $F(x_{n+1}) = F(x'_{n+1})$, we have

$$|\int_{\gamma_n^+(x)} f d\Phi_1^+ - \int_{\gamma_n^+(x')} f d\Phi_1^+| \leq C. \tag{27}$$

If C_f be the infimum of all C satisfying (27), then we norm $Lip_w^+(X)$ by setting

$$||f||_{Lip_w^+} = \sup_X f + C_f.$$

As before, let $Lip_{w,0}^+(X)$ be the subspace of $Lip_w^+(X)$ of functions whose integral with respect to ν is zero.

Take $\Phi^- \in \mathcal{Y}^-$. Any function $f \in Lip_w^+(X)$ is integrable with respect to the measure m_{Φ^-}, defined by (26), in the following sense. Let $\tilde{v} \in E^-$ be the vector corresponding to Φ^- by (20) and let $\tilde{v}^{(n)} = (Q^t)^n \tilde{v}$. Recall that

$$|\tilde{v}^{(-n)}| \to 0 \text{ exponentially fast as } n \to \infty. \tag{28}$$

Take arbitrary points $x_i^{(n)} \in X$, $n \in \mathbb{N}$ satisfying

$$F((x_i^{(n)})_n) = i, \; i = 1, \ldots, m. \tag{29}$$

and consider the expression

$$\sum_{i=1}^m \Big(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_1^+ \Big) \cdot \big(\tilde{v}^{(1-n)}\big)_i. \tag{30}$$

By (27) and (28), as $n \to \infty$ the expression (30) tends to a limit which does not depend on the particular choice of $x_i^{(n)}$ satisfying (29). This limit is denoted

$$m_{\Phi^-}(f) = \int_X f dm_{\Phi^-}.$$

Introduce a measure $\Phi_f^+ \in \mathcal{Y}^+$ by requiring that for any $\Phi^- \in \mathcal{Y}^-$ we have

$$< \Phi_f^+, \Phi^- >= \int_X f dm_{\Phi^-}. \tag{31}$$

Note that the mapping $\Xi^+ : Lip_w^+(X) \to \mathcal{Y}^+$ given by $\Xi^+(f) = \Phi_f^+$ is continuous by definition and satisfies

$$\Xi^+(f \circ \sigma) = \sigma^* \Xi^+(f). \tag{32}$$

From the definitions we also have

Proposition 8 *Let $\Phi^+(1), \dots, \Phi^+(r)$ be a basis in $\mathcal{Y}^+$ and let $\Phi^-(1), \dots, \Phi^-(r)$ be the dual basis in $\mathcal{Y}^-$ with respect to the pairing $<,>$. Then for any $f \in Lip_w^+(X)$ we have*

$$\Phi_f^+ = \sum_{i=1}^r \big(m_{\Phi^-(i)}(f)\big)\Phi^+(i).$$

2.5 Approximation.

Let Θ be a finitely-additive complex-valued measure on the semi-ring $\mathfrak{C}_0^+$. Assume that there exists a constant $\delta(\Theta)$ such that for all $x, x' \in X$ and all $n \geq 0$ we have

$$|\Theta(\gamma_n^+(x)) - \Theta(\gamma_n^+(x'))| \leq \delta(\Theta) \text{ if } F(x_{n+1}) = F(x'_{n+1}). \tag{33}$$

In this case Θ will be called a *weakly Lipschitz measure*.

Lemma 1 *There exists a constant C_Γ depending only on Γ such that the following is true. Let Θ be a weakly Lipschitz finitely-additive complex-valued measure on the semi-ring $\mathfrak{C}_0^+$. Then there exists a unique $\Phi^+ \in \mathcal{Y}_\Gamma^+$ such that for all $x \in X$ and all $n > 0$ we have*

$$|\Theta(\gamma_n^+(x)) - \Phi^+(\gamma_n^+(x))| \leq C_\Gamma \delta(\Theta) n^{m+1}. \tag{34}$$

Assign to the graph Γ the Markov compactum Y_Γ of one-sided infinite sequences of edges:

$$Y = \{y = y_1 \dots y_n \dots : y_n \in \mathcal{E}(\Gamma), F(y_{n+1}) = I(y_n)\},$$

and, as before, let σ be the shift on Y_Γ: $(\sigma y)_i = y_{i+1}$. For $y, y' \in Y_\Gamma$, write $y' \searrow y$ if $\sigma y' = y$.

Lemma 1 will be derived from

Lemma 2 *There exists a constant C_Γ depending only on Γ such that the following is true. Let φ_n be a sequence of measurable complex-valued functions on Y_Γ. Assume that there exists a constant δ such that for all $y \in Y$ and all $n \geq 0$ we have*

$$|\varphi_{n+1}(y)) - \sum_{y' \searrow y} \varphi_n(y')| \leq \delta \tag{35}$$

and for all $n \geq 0$ and all $y, \tilde{y} \in Y_\Gamma$ satisfying $F(y_1) = F(\tilde{y}_1)$ we have

$$|\varphi_n(y)) - \varphi_n(\tilde{y})| \leq \delta. \tag{36}$$

Then there exists a unique $v \in E^+$ such that for all $y \in Y$ and all $n > 0$ we have

$$|\varphi_n(y)) - (Q^n v)_{F(y_{n+1})}| \leq C_\Gamma \delta n^{m+1}. \tag{37}$$

Proof of Lemma 2. Take arbitrary points $y(i) \in Y_\Gamma$ in such a way that

$$F(y(i)_1) = i.$$

Introduce a sequence of vectors $v(n) \in \mathbb{C}^m$ by the formula

$$v(n)_i = \varphi_n(y(i)).$$

From (36) for any $y \in Y$ we have

$$|\varphi_n(y) - v(n)_{F(y_1)}| \leq \delta,$$

and from (35), (36) we have

$$|Qv(n) - v(n+1)| \leq \delta \cdot ||Q||.$$

To prove Lemma 2, it suffices now to establish the following

Proposition 9 *Let V be a finite-dimensional complex linear space, let $S : V \to V$ be a linear operator and let $V^+ \subset V$ be the subspace spanned by vectors corresponding to Jordan cells of S with eigenvalues exceeding 1 in absolute value. There exists a constant $C > 0$ depending only on S such that the following is true. Assume that the vectors $v(n) \in V$, $n \in \mathbb{N}$, satisfy*

$$|Sv(n) - v(n+1)| < \delta$$

for all $n \in \mathbb{N}$ and some constant $\delta > 0$. Then there exists a unique $v \in V^+$ such that for all $n \in \mathbb{N}$ we have

$$|S^n v - v(n)| \leq C \cdot \delta \cdot n^{\dim V - \dim V^+ + 1}. \tag{38}$$

Proof of Proposition 9. By definition, the subspace V^+ is S-invariant and S is invertible on V^+; we have furthermore that $|Q^{-n}v| \to 0$ exponentially fast as $n \to \infty$. Let V^- be the subspace spanned by Jordan cells corresponding to eigenvalues of absolute value at most 1; for $v \in V^-$, we have $|Q^n v| < Cn^{\dim V - \dim V^+}$ as $n \to \infty$. We have the decomposition $V = V^+ \oplus V^-$. Let

$$u(0) = v(0), u(n+1) = v(n+1) - Sv(n).$$

Decompose $u(n) = u^+(n) + u^-(n)$, where $u^+(n) \in V^+$, $u^-(n) \in V^-$. Denote

$$v^+(n+1) = u^+(n+1) + Su^+(n) + \cdots + S^n u^+(1);$$

$$v^-(n+1) = u^-(n+1) + Su^-(n) + \cdots + S^n u^-(1);$$
$$v = u^+(0) + S^{-1}u^+(1) + \cdots + S^{-n}u^+(n) + \ldots .$$

By definition, $|v^-(n+1)|$ is bounded above by $C\delta n^{\dim V - \dim V^+ + 1}$ and there exists $\tilde{C}$ such that $|S^n v - v^+(n)| < \tilde{C}\delta$ for all $n \in \mathbb{N}$, whence (38) follows. Uniqueness of v follows from the fact that for any nonzero $v' \in V^+$ the sequence $|S^n v'|$ grows exponentially as $n \to \infty$. Proposition 9 and Lemmas 1, 2 are proved completely.

Let $f \in Lip_w^+(X)$. We then have a measure Θ_f on the semi-ring $\mathfrak{C}_0^+$ given, for $\gamma \in \mathfrak{C}_0^+$, by the formula

$$\Theta_f(\gamma) = \int_\gamma f d\Phi_1^+.$$

By (27), the measure Θ_f satisfies the assumptions of Lemma 1. Let $\Xi_f^+ \in \mathfrak{Y}^+$ be the measure assigned to Θ_f by Lemma 1.

Lemma 3 *Let $f \in Lip_w^+(X)$, $\Phi^- \in \mathfrak{Y}_\Gamma^-$. Then*

$$< \Xi_f^+, \Phi^- >= \int_X f dm_{\Phi^-}. \tag{39}$$

Proof: Choose the points $x_i^{(n)} \in X$ satisfying (29). As above, let $\tilde{v} \in E^-$ be the vector corresponding to Φ^- by (20) and let $\tilde{v}^{(n)} = (Q^t)^n \tilde{v}$, $n \in \mathbb{Z}$. For any $\varepsilon > 0$ and $n > 0$ sufficiently large, by definition, we have

$$\big| m_{\Phi^-}(f) - \sum_{i=1}^m \big(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_1^+ \big) \cdot \big(\tilde{v}^{(-n)}\big)_i \big| < \varepsilon. \tag{40}$$

By definition of Ξ_f^+ and Lemma 1 we have

$$|\sum_{i=1}^m \big(\int_{\gamma_n^+(x_i^{(n)})} f d\Phi_1^+ \big) \cdot \big(\tilde{v}^{(-n)}\big)_i - \sum_{i=1}^m \big(\Xi_f^+(\gamma_n^+(x_i^{(n)})\big) \cdot \big(\tilde{v}^{(-n)}\big)_i | < C_\Gamma \cdot n^{m+1} |\tilde{v}_i^{(-n)}|,$$

and, by (28), the right-hand side tends to 0 exponentially fast as $n \to \infty$.

It remains to notice that, by definition,

$$\sum_{i=1}^m \big(\Xi_f^+(\gamma_n^+(x_i^{(n)})\big) \cdot \big(\tilde{v}^{(-n)}\big)_i =< \Xi_f^+, \Phi^- >,$$

and the Lemma is proved completely.

We have thus established that $\Xi_f^+ = \Phi_f^+$, where Φ_f^+ is given by (31).

2.6 Orderings.

Following S. Ito [7], A.M. Vershik [15, 16], assume that a partial order $\mathfrak{o}$ is given on $\mathcal{E}(\Gamma)$ in such a way that edges starting at a given vertex are ordered linearly, while edges starting at different vertices are not comparable. An edge will be called *maximal* (with respect to $\mathfrak{o}$) if there does not exist a greater edge; *minimal*, if there does not exist a smaller edge; and an edge e will be called *the successor* of e' if $e > e'$ but there does not exist e'' such that $e > e'' > e'$.

The ordering $\mathfrak{o}$ is extended to a partial ordering of X_Γ: we write $x < x'$ if there exists $l \in \mathbb{Z}$ such that $x_l < x'_l$ and $x_n = x'_n$ for all $n > l$. Under this ordering each leaf γ^+_∞ of the foliation $\mathcal{F}^+$ is linearly ordered, while points lying on different leaves are not comparable.

Let $Max(\mathfrak{o})$ be the set of points $x \in X$, $x = (x_n)_{n\in\mathbb{Z}}$, such that each x_n is a maximal edge. Similarly, $Min(\mathfrak{o})$ denotes the set of points $x \in X$, $x = (x_n)_{n\in\mathbb{Z}}$, such that each x_n is a minimal edge. Since edges starting at a given vertex are ordered linearly, the cardinalities of $Max(\mathfrak{o})$ and $Min(\mathfrak{o})$ do not exceed m.

If a leaf γ^+_∞ does not intersect $Max(\mathfrak{o})$, then it does not have a maximal element; similarly, if γ^+_∞ does not intersect $Min(\mathfrak{o})$, then it does not have a minimal element.

For $x(1), x(2) \in \gamma^+_\infty$, let

$$[x(1), x(2)] = \{x' \in \gamma^+_\infty : x(1) \le x' \le x(2)\}.$$

The sets $(x(1), x(2)]$, $[x(1), x(2))$, $(x(1), x(2))$ are defined similarly.

Proposition 10 *Let $x \in X$. If $\gamma^+_\infty(x) \cap Max(\mathfrak{o}) = \emptyset$, then for any $t \ge 0$ there exists a point $x' \in \gamma^+_\infty(x)$ such that*

$$\Phi^+_1([x, x']) = t. \tag{41}$$

Proof. Let $V(x) = \{t : \exists x' \ge x : \Phi^+_1([x, x']) = t\}$. Since $\gamma^+_\infty(x) \cap Max(\mathfrak{o}) = \emptyset$, for any n there exists $x'' \in \gamma^+_\infty(x)$ such that all points in $\gamma^+_n(x'')$ are greater than x. Since $\Phi^+_1(\gamma^+_n(x''))$ grows exponentially, uniformly in x'', as $n \to \infty$, the set $V(x)$ is unbounded. Furthermore, since $\Phi^+_1(\gamma^+_n(x''))$ decays exponentially, uniformly in x'', as $n \to -\infty$, the set $V(x)$ is dense in $\mathbb{R}_+$. Finally, by compactness of X, the set $V(x)$ is closed, which concludes the proof of the Proposition.

A similar proposition, proved in the same way, holds for negative t.

Proposition 11 *Let $x \in X$. If $\gamma^+_\infty(x) \cap Min(\mathfrak{o}) = \emptyset$, then for any $t \ge 0$ there exists a point $x' \in \gamma^+_\infty(x)$ such that*

$$\Phi^+_1([x', x]) = t. \tag{42}$$

Define an equivalence relation $\sim$ on X by writing $x \sim x'$ if $x \in \gamma^+_\infty(x')$ and $\Phi^+_1([x, x']) = \Phi^+_1([x', x]) = 0$. The equivalence classes admit the following explicit description, which is clear from the definitions.

Proposition 12 *Let $x, x' \in X$ be such that $x \in \gamma^+_\infty(x')$, $x < x'$ and $\Phi^+_1([x, x']) = 0$. Then there exists $n \in \mathbb{Z}$ such that*

1. *x'_n is a successor of x_n;*
2. *x is the maximal element in $\gamma_n(x)$;*
3. *x' is the minimal element in $\gamma_n(x')$.*

In other words, $\Phi^+_1([x, x']) = 0$ if and only if $(x, x') = \emptyset$. In particular, equivalence classes consist at most of two points and, ν-almost surely, of only one point.

Denote $X_{\mathfrak{o}} = X/\sim$, let $\pi_{\mathfrak{o}} : X \to X_{\mathfrak{o}}$ be the projection map and set $\nu_{\mathfrak{o}} = (\pi_{\mathfrak{o}})_* \nu$. The probability spaces $(X_{\mathfrak{o}}, \nu_{\mathfrak{o}})$ and (X, ν) are measurably isomorphic; in what follows, we shall often omit the index $\mathfrak{o}$. The foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ descend to the space $X_{\mathfrak{o}}$; we shall denote their images on $X_{\mathfrak{o}}$ by the same letters and, as before, denote by $\gamma^+_\infty(x)$, $\gamma^-_\infty(x)$ the leaves containing $x \in X_{\mathfrak{o}}$.

Now let $x \in X_{\mathfrak{o}}$ satisfy $\gamma^+_\infty(x) \cap Max(\mathfrak{o}) = \emptyset$. By Proposition 10, for any $t \geq 0$ there exists a unique x' satisfying (41). Denote $h^+_t(x) = x'$. Similarly, if $x \in X_{\mathfrak{o}}$ satisfy $\gamma^+_\infty(x) \cap Min(\mathfrak{o}) = \emptyset$. By Proposition 11, for any $t \geq 0$ there exists a unique x' satisfying (42). Denote $h^+_{-t}(x) = x'$.

We thus obtain a flow h^+_t, which is well-defined on the set

$$X_{\mathfrak{o}} \setminus \Big(\bigcup_{x \in Max(\mathfrak{o}) \cup Min(\mathfrak{o})} \gamma^+_\infty(x) \Big),$$

and, in particular, ν-almost surely on $X_{\mathfrak{o}}$. By (16), the flow h^+_t preserves the measure ν.

More generally, it is clear from the definitions that for any $\Phi^- \in \mathcal{Y}^-$, the measure m_{Φ^-}, defined by (26), satisfies

$$(h^+_t)_* m_{\Phi^-} = m_{\Phi^-},$$

similarly to G. Forni's invariant distributions [5], [6].

Remark. S.Ito in [7] gives a construction of a flow similar to the one above. The flow h^+_t is a continuous-time analogue of a Vershik automorphism [15] (of which a variant also occurs in Ito's work [7]), and, in fact, is a suspension flow over the corresponding Vershik's automorphism, a point of view adopted in [4].

2.7 Decomposition of Arcs.

We assume that an ordering $\mathfrak{o}$ is fixed on Γ. Denote by $\mathfrak{C}(\mathfrak{o})$ the semi-ring of subsets of X_Γ of the form $[x, x')$, where $x < x'$. Any measure $\Phi^+ \in \mathcal{Y}^+$ can be extended to $\mathfrak{C}(\mathfrak{o})$ in the following way.

Let $\mathfrak{R}^+_n$ be the ring generated by the semi-ring $\mathfrak{C}^+_n$. For $\gamma \in \mathfrak{C}(\mathfrak{o})$, denote by $\gamma(n)$ the smallest (by inclusion) element of the ring $\mathfrak{R}^+_{-n}$ containing γ and

let $\hat{\gamma}(n)$ be the greatest (by inclusion) element of the ring $\mathfrak{R}^+_{-n}$ contained in γ (possibly, $\hat{\gamma}(n) = \emptyset$). By definition,

$$\hat{\gamma}(n) \subset \hat{\gamma}(n+1) \subset \gamma(n+1) \subset \gamma(n);$$

$$\gamma(n) \setminus \hat{\gamma}(n) = \bigsqcup_{i=1}^{l_n} \gamma_i^{(n)}, \tag{43}$$

where $\gamma_i^{(n)} \in \mathfrak{C}^+_{-n}$, $l_n \le ||Q||$, and

$$\gamma(n) \setminus \gamma(n+1) = \bigsqcup_{i=1}^{L_n} \gamma_i^{(n+1)}, \tag{44}$$

where $\gamma_i^{(n+1)} \in \mathfrak{C}^+_{-n-1}$, $L_n \le 2||Q||$.

By definition, if $\Phi^+ \in \mathcal{Y}^+$, then there are only m possible values of $\Phi^+(\gamma)$ for $\gamma \in \mathfrak{C}^+_{-n}$, and the maximum of these decays exponentially as $n \to \infty$. We thus have

Proposition 13 *There exists positive constants C_Γ, depending only on Γ, such that the following is true. Let $v_0 = 0$, $v_1, \dots, v_l \in E^+$, $Qv_i = \exp(\theta)v_i + v_{i-1}$. Assume $v \in \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_l$ satisfies $|v| = 1$ and let $\Phi^+_v = \mathcal{I}_\Gamma(v)$. Then for any $\gamma \in \mathfrak{C}(\mathfrak{o})$ we have*

$$|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \le C_\Gamma n^{l-1} \exp(-(\Re\theta)n);$$

$$|\Phi^+_v(\hat{\gamma}(n)) - \Phi^+_v(\hat{\gamma}(n+1))| \le C_\Gamma n^{l-1} \exp(-(\Re\theta)n).$$

decay exponentially as $n \to \infty$. In particular, if $v \in E^+$, $Qv = \exp(\theta)v$, $|v| = 1$, then

$$|\Phi^+_v(\gamma(n)) - \Phi^+_v(\gamma(n+1))| \le C_\Gamma \exp(-(\Re\theta)n);$$

$$|\Phi^+_v(\hat{\gamma}(n)) - \Phi^+_v(\hat{\gamma}(n+1))| \le C_\Gamma \exp(-(\Re\theta)n).$$

Consequently, for any $\Phi^+ \in \mathcal{Y}^+$, $\gamma \in \mathfrak{C}(\mathfrak{o})$, the sequence $\Phi^+(\gamma(n))$ converges as $n \to \infty$, and we set

$$\Phi^+(\gamma) = \lim_{n\to\infty} \Phi^+(\gamma(n)).$$

By (43), we also have

$$\Phi^+(\gamma) = \lim_{n\to\infty} \Phi^+(\hat{\gamma}(n)).$$

Proposition 14 *The measure Φ^+ is finitely-additive on $\mathfrak{C}(\mathfrak{o})$.*

Proof: Let $v \in E^+$ be such that $\Phi^+ = \Phi^+_v$ and let $\gamma_0, \gamma_1, \dots, \gamma_k \in \mathfrak{C}(\mathfrak{o})$ satisfy

$$\gamma_0 = \bigsqcup_{i=1}^{k} \gamma_i.$$

Consider the arcs $\gamma_0(n), \gamma_1(n), \ldots, \gamma_k(n)$. We have

$$\gamma_0(n) \subset \bigcup_{i=1}^{k} \gamma_i(n). \tag{45}$$

and decompose

$$\gamma_i(n) = \bigsqcup \gamma_{ij}(n+1),$$

where $\gamma_{ij}(n+1) \in \mathfrak{C}^+_{-n-1}$.

By (45), each of the arcs $\gamma_{0j}(n+1)$ is also encountered among the arcs $\gamma_{ij}(n+1)$ (possibly, more than once, but not more than k times). Consider the collection $\gamma_{ij}(n+1)$ and cross out all the arcs $\gamma_{0j}(n+1)$; by maximality, and since our ordering is linear on each leaf of the foliation $\mathcal{F}^+$, there will remain not more than $2k||Q||$ arcs, whence we obtain

$$\big|\sum_{i=1}^{k} \Phi^+(\gamma_i(n)) - \Phi^+(\gamma_0(n))\big| \leq 2k||Q|| \cdot |Q^{-n-1}v|,$$

and, since the right-hand side decays exponentially as $n \to \infty$, the Proposition is proved.

Lemma 4 *There exists a constant C_Γ depending only on Γ such that the following is true. Let $f \in Lip^+_w(X_\Gamma)$ and let $\Phi^+_f \in \mathcal{Y}^+$ be given by (31). For any $\gamma \in \mathfrak{C}(\mathfrak{o})$ we have*

$$\big|\int_\gamma f d\Phi^+_1 - \Phi^+_f(\gamma)\big| \leq C_\Gamma ||f||_{Lip^+_w}(1 + \log(1 + \Phi^+_1(\gamma))^{m+1}. \tag{46}$$

Indeed, for $\gamma \in \mathfrak{C}^+$ this follows from Lemma 1, and for all other arcs from Proposition 13.

2.8 Ergodic averages of the flow h^+_t.

Let $\Phi^+ \in \mathcal{Y}^+$ and denote $\Phi^+[x,t] = \Phi^+_i([x, h^+_t x])$. The function $\Phi^+(x,t)$ is an additive cocycle over the flow h^+_t. Let $f \in Lip^+_w(X_\Gamma)$, and let Φ^+_f be defined by (31). By definition, $\Phi_{f\circ h^+_t} = \Phi^+_f$; recall from (32) that $\Phi^+_{f\circ\sigma} = \sigma^*\Phi^+_f$. Lemma 4 implies

Theorem 5 *There exists a positive constant C_Γ depending only on Γ such that for any $f \in Lip^+_w(X_\Gamma)$, for all $x \in X$ and all $T > 0$ we have*

$$\big|\int_0^T f \circ h^+_t(x)dt - \Phi^+_f(x,t)\big| \leq C_\Gamma ||f||_{Lip}(1 + \log(1+T))^{m+1}.$$

Given a bounded measurable function $f : X \to \mathbb{R}$ and $x \in X$, introduce a continuous function $\mathfrak{S}_n[f,x]$ on the unit interval by the formula

$$\mathfrak{S}_n[f,x](\tau) = \int\limits_0^{\tau\exp(n\theta_1)} f \circ h^+_t(x)dt. \tag{47}$$

The functions $\mathfrak{S}_n[f,x]$ are $C[0,1]$-valued random variable on the probability space (X_Γ, ν_Γ).

Theorem 6 *If Q has a simple real second eigenvalue* $\exp(\theta_2)$*, $\theta_2 > 0$, then there exists a continuous functional $\alpha : Lip_w^+(X) \to \mathbb{R}$ and a compactly supported non-degenerate measure η on $C[0,1]$ such that for any $f \in Lip_{w,0}^+(X)$ satisfying $\alpha(f) \neq 0$ the sequence of random variables*

$$\frac{\mathfrak{S}_n[f,x]}{\alpha(f)\exp(n\theta_2)}$$

converges in distribution to η as $n \to \infty$.

Remark. Compactness of the support of η is understood in the sense of the Tchebycheff topology on $C[0,1]$. Nondegeneracy of the measure η means that if $\varphi \in C[0,1]$ is distributed according to η, then for any $t_0 \in (0,1]$ the distribution of the real-valued random variable $\varphi(t_0)$ is not concentrated at a single point.

The measure η is constructed as follows: let v_2 be an eigenvector with eigenvalue $\exp(\theta_2)$, set $\Phi_2^+ = \mathcal{I}(v_2)$ (see (19)); then η is the distribution of $\Phi_2^+(x,\tau)$, $0 \le \tau \le 1$, considered as a $C[0,1]$-valued random variable on the space $X_\Gamma, \nu_\Gamma)$. The functional $\alpha(f)$ is constructed as follows: under the assumptions of Theorem 6, the matrix Q^t also has the simple real second eigenvalue $\exp(\theta_2)$; let $\tilde{v}_2$ be the eigenvector with eigenvalue $\exp(\theta_2)$, normalized in such a way that $\sum_{i=1}^m (v_2)_i(\tilde{v}_2)_i = 1$; set $\Phi_2^- = \tilde{\mathcal{I}}(\tilde{v}_2)$ (see (21)),and let $m_{\Phi_2^-}$ be given by (26); then

$$\alpha(f) = \int f dm_{\Phi_2^-}.$$

2.9 The diagonalizable case.

As an illustration, consider the case when $Q|_{E^+}$ is diagonalizable with eigenvalues $\exp(\theta_i)$, $i = 1, \ldots, r$, $\Re(\theta_i) > 0$. The Perron-Frobenius vector h corresponds to $\exp(\theta_1)$; let $v_2, \ldots, v_r$ be eigenvectors corresponding to $\exp(\theta_i)$: thus $Qv_i = \exp(\theta_i)v_i$, $i = 2, \ldots, r$ and

$$E^+ = \mathbb{C}h \oplus \mathbb{C}v_2 \oplus \cdots \oplus \mathbb{C}v_r$$

We have a similar direct-sum representation for Q^t:

$$\tilde{E}^+ = \mathbb{C}\lambda \oplus \mathbb{C}\tilde{v}_2 \oplus \cdots \oplus \mathbb{C}\tilde{v}_r,$$

where $Q^t\tilde{v}_i = \exp(\theta_i)\tilde{v}_i$, $i = 2, \ldots, r$. For $i \neq j$ we have

$$\sum_{l=1}^m (v_i)_l(\tilde{v}_j)_l = 0, \tag{48}$$

and, for normalization, let us assume that for all $i = 1, \ldots, r$ we have

$$\sum_{l=1}^m (v_i)_l(\tilde{v}_i)_l = 1. \tag{49}$$

Let $\Phi_i^+ = \mathcal{I}(v_i)$, $\Phi_i^- = \tilde{\mathcal{I}}(\tilde{v}_i)$, $i = 2, \ldots, r$. Since $\Phi_1^+ = \mathcal{I}(h)$, the measures Φ_i^+, $i = 1, \ldots, r$, form a basis in $\mathcal{Y}^+$, for which the measures $\Phi_1^- = \tilde{\mathcal{I}}(\lambda)$, $\Phi_2^-, \ldots, \Phi_r^-$ form a dual basis in $\mathcal{Y}^-$.

For $i = 1, \ldots, r$, from (26) we have the measures $m_{\Phi_i^-} = \Phi_1^+ \times \Phi_i^-$. For instance, $m_{\Phi_1^-} = \nu$. Theorem 5 now implies

Corollary 1 *For any $f \in Lip_w^+(X_\Gamma)$ we have*

$$\Big|\int_0^T f \circ h_t^+(x)dt - T\int_X f d\nu - \sum_{i=2}^r \Phi_i^+(x,T)\big(m_{\Phi_i^-}(f)\big)\Big| \le C_\Gamma ||f||_{Lip}(1+\log(1+T))^{m+1},$$

where C_Γ is a constant depending only on Γ.

For the action of the shift we have:

$$(\sigma)_*\Phi_i^+ = \exp(-\theta_i)\Phi_i^+, \ i = 1, \ldots, r; \tag{50}$$

$$(\sigma)_*\Phi_i^- = \exp(\theta_i)\Phi_i^-, \ i = 1, \ldots, r. \tag{51}$$

Corollary 1 now yields

$$\int_0^{\tau\exp(\theta_1 n)} f \circ h_t^+(x)dt = \sum_{i=1}^r \exp(n\theta_i) m_{\Phi_i^-}(f)\Phi_i^+(\sigma^n x, \tau) + O(n^{m+1}). \tag{52}$$

2.10 The Hölder property.

As above, we write $\Phi^+(x,t) = \Phi^+([x, h_t^+ x])$. Our next aim is to show that $\Phi^+(x,t)$ is Hölder in t for any $x \in X_{\mathfrak{o}}$.

Proposition 15 *There exist positive constants C_Γ and t_0, depending only on Γ such that the following is true. Let $v \in E^+$, $Qv = \exp(\theta)v$, $|v| = 1$. Then for all $x \in X$ and positive $t < t_0$ we have*

$$|\Phi_v^+(x,t)| \le C_\Gamma t^{\Re\theta/\theta_1}.$$

Proposition 16 *There exist positive constants C_Γ and t_0, depending only on Γ such that the following is true. Let $v_0 = 0$, $v_1, \ldots, v_l \in E^+$, $Qv_i = \exp(\theta)v_i + v_{i-1}$. Assume $v \in \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_l$ satisfies $|v| = 1$. Then for all $x \in X$ and positive $t < t_0$ we have*

$$|\Phi_v^+(x,t)| \le C_\Gamma |\log t|^{l-1} t^{\Re\theta/\theta_1}.$$

Proof of Propositions 15, 16. Denote $\gamma = [x, h_t^+ x]$. If t is small enough, then $\hat{\gamma}(0) = \emptyset$. Let n_0 be the smallest positive integer such that $\hat{\gamma}(n_0) \neq \emptyset$. There exist positive constants C_1, C_2, depending only on Γ, such that

$$C_1 t \le \exp(-\theta_1 n_0) \le C_2 t,$$

and Propositions 15, 16 follow now from Proposition 13.

Corollary 2 *There exist positive constants $\theta > 0$ and $t_0 > 0$ depending only on Q such that for all $v \in E^+$, $|v| = 1$, all $x \in X$ and all positive $t < t_0$ we have*

$$|\Phi_v^+(x,t)| \le t^{\theta/\theta_1}.$$

For $v \in E^+$, $|v| = 1$ denote

$$\theta_v = \lim_{n\to\infty} \frac{\log |Q^n v|}{n}.$$

Corollary 3 *For any $\varepsilon > 0$ there exists a constant T_ε depending only on ε and Γ such that for any $v \in E^+$, $|v| = 1$, any $x \in X$ and any $T > T_\varepsilon$, we have*

$$|\Phi_v^+(x,T)| \le T^{\theta_v/\theta_1+\varepsilon}.$$

Proof: Indeed, let t_0 be the constant given by Proposition16. Let $n_0 = n_0(T)$ be the smallest such integer that $T = \tau \exp(n(T)\theta_1)$, where $\tau < t_0$. Since $\Phi_v^+(x,T) = \Phi_{Q^n v}^+(\sigma^n x, \tau)$ for all n, it follows from Proposition 16 that

$$|\Phi^+(x,T)| \le C_\Gamma n_0^{m+1} \exp(n_0 \Re(\theta_v)) \le C_\Gamma T^{\theta_v/\theta_1+\varepsilon}$$

if T is sufficiently large (depending only on ε).

Corollary 4 *For any $v \in E^+$ we have*

$$\limsup_{T\to\infty} \frac{\log |\Phi_v^+(x,T)|}{\log T} = \frac{\theta_v}{\theta_1}. \tag{53}$$

Indeed, the upper bound for the limit superior follows from Corollary 3, and the lower bound is immediate from the relation $\Phi_v^+(\gamma_n(x)) = (Q^n v)_{F(x_{n+1})}$.

Corollary 5 *For any $\tau \in \mathbb{R}$ and any $v \in E^+$ satisfying $v \ne 0$, $\sum_{i=1}^{m} v_i \lambda_i = 0$, the function $\Phi_v^+(x,\tau)$ is not a constant in x.*

Proof: Indeed, assume $\Phi_v^+(x,\tau) = c$ identically. Then $\Phi^+(x, k\tau) = kc$, which contradicts (53): is $c = 0$, then the limit superior is 0; if $c \ne 0$, then the limit superior is 1.

2.11 Tightness.

In this subsection, we assume that Q has a simple real second eigenvalue $\exp(\theta_2)$, $\theta_2 > 0$. Let v_2 be the corresponding eigenvector and let $\Phi_2^+ = \mathcal{I}(v_2)$. Take $x \in X$ and consider $\Phi^+(x,\tau)$ as a continuous function of τ on the unit interval. Let η be the distribution of $\Phi_2^+(x,\tau)$ in $C[0,1]$. Note that by Corollary 5, for any τ_0 the value of $\Phi_2^+(x,\tau)$ is not constant on X, so the measure η is nondegenerate.

Let $\mathfrak{S}_n[f,x]$ be defined by the equation (47). Introduce a sequence of measures μ_n on $C[0,1]$ by the formula $\mu_n = \mathfrak{S}[n,f]_* \nu_\Gamma$.

By Theorem 8.1 in Billingsley [3], p.54, to prove Theorem 6 it suffices to establish the following two Lemmas.

Lemma 5 *Finite-dimensional distributions of the measures μ_n weakly converge to those of η.*

Lemma 6 *The family μ_n is tight in $C[0,1]$.*

Proof of Lemma 5. By Theorem 5

$$\int_0^T f \circ h_t^+(x)dt = \Phi_f^+(x,T) + O((\log T)^{m+1}).$$

Let v_2 be the eigenvector corresponding to the eigenvalue $\exp(\theta_2)$, $|v| = 1$, and let $\Phi_2^+ \in \mathcal{Y}^+$ be the corresponding measure. We have

$$E^+ = \mathbb{C}v_2 \oplus E_3,$$

where E_3 is spanned by Jordan cells corresponding to eigenvalues with absolute value less than $\exp(\theta_2)$. Let ζ be a number smaller than θ_2 but greater than the spectral radius of $Q|_{E_3}$. Write

$$\Phi_f^+ = \alpha(f)\Phi_2^+ + \beta(f)\Phi_{v_3}^+, \tag{54}$$

where $v_3 \in E^+$, $|v_3| = 1$, and $\alpha(f), \beta(f)$ are continuous functionals on $Lip_w^+(X)$, so, in particular, we have

$$|\alpha(f)| < C_{01}||f||_{Lip_w^+};\ |\beta(f)| < C_{02}||f||_{Lip_w^+},$$

where the constants C_{01}, C_{02} only depend on Γ.

By Corollary 2, there exists t_0 depending only on Γ such that for any positive t such that $t < t_0$, any $x \in X$ and any $v \in E^+$ satisfying $|v| = 1$ we have

$$|\Phi_v^+(x,t)| \leq 1. \tag{55}$$

Write $T = t\exp(n\theta_1)$, where $t < t_0$. Since $\Phi_{v_3}^+(x,T) = \Phi_{Q^n v_3}^+(\sigma^n x, t)$, for all sufficiently large n, we have $|Q^n v_3| < \exp(\zeta n)$ and therefore

$$|\Phi_{v_3}^+(x, \tau\exp(n\theta_1))| < \exp(n\zeta) \tag{56}$$

for all $x \in X$. By Theorem 5 we have

$$\Big| \int_0^{\tau\exp(n\theta_1)} f \circ h_t^+(x)dt - \Phi_f^+(x, \tau\exp(\theta_1 n))\Big| = O(n^{m+1}). \tag{57}$$

Since

$$\Phi_f^+(x, \tau\exp(n\theta_1)) = \alpha(f)\Phi_2^+((x, \tau\exp(n\theta_1)) + \beta(f)\Phi_{v_3}^+(x, \tau\exp(n\theta_1))$$

combining the equality

$$\Phi_2^+(x, \tau\exp(n\theta_1)) = \exp(n\theta_2)\Phi_2^+(\sigma^n x, \tau)$$

with the bound (56), we obtain, for all large n and all $x \in X$, uniformly in $\tau \in [0,1]$, the estimate

$$|\mathfrak{S}_n[f,x](\tau) - \alpha(f)\Phi_2^+(\sigma^n x, \tau)| \leq C_\Gamma ||f||_{Lip_w^+} \exp((\zeta - \theta_2)n).$$

Since σ preserves the measure ν, it follows that the k-dimensional distributions of $\big(\mathfrak{S}_n[f,x](\tau_1), \mathfrak{S}_n[f,x](\tau_2), \ldots, \mathfrak{S}_n[f,x](\tau_k)\big)$ converge to the k-dimensional distribution of $\big(\Phi_2^+(x,\tau_1), \Phi_2^+(x,\tau_2), \ldots, \Phi_2^+(x,\tau_k)\big)$, and Lemma 5 is proved.

The argument above yields also

Proposition 17 *There exist positive constants $C_0 = C_0(\Gamma)$ and $T_0 = T_0(\Gamma)$ such that for any $x \in X$, any $f \in Lip_{w,0}^+(X)$ and any $T > T_0$ we have*

$$|\int_0^T f \circ h_t^+(x)dt| \leq C_0 \cdot ||f||_{Lip_w^+} \cdot T^{\theta_2/\theta_1}.$$

Indeed, for sufficiently large T, $T = t\exp(n\theta_1)$, where $t < t_0$, from (54) we have

$$\Phi_f^+(x,T) = \alpha(f)\exp(n\theta_2)\Phi_2^+(\sigma^n x, t) + O(\exp(n\zeta)).$$

Since, by (55), we have $|\Phi_2^+(\sigma^n x, t)| \leq 1$, Proposition 17 is established.

We proceed to the proof of Lemma 6.

Proposition 18 *There exists a constant C_Γ depending only on Γ such that for any $f \in Lip_{w,0}^+(X)$, any $n > 0$, any $x \in X$ and any $\tau_1, \tau_2 \in [0,1]$, we have*

$$|\mathfrak{S}_n[x,f](\tau_2) - \mathfrak{S}_n[x,f](\tau_1)| \leq C_\Gamma ||f||_{Lip_w^+} |\tau_2 - \tau_1|^{\theta_2/\theta_1}.$$

Lemma 6 follows from Proposition 18 by the Arzelà-Ascoli Theorem.

Proof of Proposition 18: Let $\tau_1, \tau_2 \in [0,1]$, $\tau_1 < \tau_2$. For brevity, write $\mathfrak{S}_n = \mathfrak{S}_n[f,x]$. We have then

$$\mathfrak{S}_n(\tau_2) - \mathfrak{S}_n(\tau_1) = \frac{1}{\exp(n\theta_2)} \int\limits_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h_t^+(x)dt.$$

Let T_0 be the constant given by Proposition 17 and assume first that

$$(\tau_2 - \tau_1) \cdot \exp(n\theta_1) \geq T_0.$$

By Proposition 17 we have

$$\int\limits_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h_t(x)dt \leq C||f||_{Lip_w^+} \cdot (\tau_2 - \tau_1)^{\theta_2/\theta_1} \exp(n\theta_2),$$

and, consequently,

$$|\mathfrak{S}_n(\tau_2) - \mathfrak{S}_n(\tau_1)| \leq C_{33}(\tau_2 - \tau_1)^{\theta_2/\theta_1},$$

where the constant C_{33} only depends on Γ.

Now let $\tau_2 - \tau_1 = \tau_0 \exp(-n\theta_1)$, $\tau_0 < T_0$. Since

$$\exp(-n\theta_2) = ((\tau_2 - \tau_1)/\tau_0)^{\theta_2/\theta_1},$$

using boundedness of f, write

$$\frac{1}{\exp(n\theta_2)} \int_{\tau_1 \exp(n\theta_1)}^{\tau_2 \exp(n\theta_1)} f \circ h_t^+(x) dt \leq \exp(-n\theta_2) \cdot ||f||_\infty \cdot \tau_0 \leq$$

$$\leq \tau_0^{1-\theta_2/\theta_1} ||f||_\infty (\tau_2 - \tau_1)^{\theta_2/\theta_1} \leq T_0^{1-\theta_2/\theta_1} ||f||_\infty (\tau_2 - \tau_1)^{\theta_2/\theta_1},$$

and the Proposition is proved. Theorem 6 is proved completely.

2.12 A symbolic coding for translation flows on surfaces.

To derive Theorems 1, 2 from Theorems 5, 6, it remains to observe that the vertical flow on the stable foliation of a pseudo-Anosov diffeomorphism is isomorphic to a symbolic flow on the asymptotic foliation of a Markov compactum obtained from the decomposition of the underlying surface into Veech's zippered rectangles, see [4], Sec. 4. The identification of E^+ (and, consequently, of $\mathcal{Y}^+$) with the corresponding subspace in cohomology is given by Proposition 4.16 in Veech[14]. The fact that the pairing between cocycles corresponds to the cup-product is immediate from Proposition 4.19 in [14].

3 Spaces of Markov Compacta.

Let $\mathfrak{G}$ be the set of all oriented graphs on m vertices such that there is an edge starting at every vertex and an edge ending at every vertex. As before, for a graph $\Gamma \in \mathfrak{G}$, we denote by $\mathcal{E}(\Gamma)$ the set of its edges and by $A(\Gamma)$ its incidence matrix: $A_{ij}(\Gamma) = \#\{e \in \mathcal{E}(\Gamma) : I(e) = i, F(e) = j\}$. Denote $\Omega = \mathfrak{G}^{\mathbb{Z}}$:

$$\Omega = \{\omega = \dots \omega_{-n} \dots \omega_n \dots, \omega_i \in \mathfrak{G}, i \in \mathbb{Z}\},$$

For $\omega \in \Omega$, denote by $X(\omega)$ the corresponding Markov compactum:

$$X(\omega) = \{x = \dots x_{-n} \dots x_n \dots, x_n \in \mathcal{E}(\omega_n), F(x_{n+1}) = I(x_n)\}.$$

For $x \in X$, $n \in \mathbb{Z}$, introduce the sets

$$\gamma_n^+(x) = \{x' \in X(\omega) : x'_t = x_t, t \geq n\}; \ \gamma_n^-(x) = \{x' \in X(\omega) : x'_t = x_t, t \leq n\};$$

$$\gamma_\infty^+(x) = \bigcup_{n \in \mathbb{Z}} \gamma_n^+(x); \ \gamma_\infty^-(x) = \bigcup_{n \in \mathbb{Z}} \gamma_n^-(x).$$

The sets $\gamma_\infty^+(x)$ are leaves of the asymptotic foliation $\mathcal{F}_\omega^+$ on $X(\omega)$; the sets $\gamma_\infty^-(x)$ are leaves of the asymptotic foliation $\mathcal{F}_\omega^-$ on $X(\omega)$.

For $n \in \mathbb{Z}$ let $\mathfrak{C}^+_{n,\omega}$ be the collection of all subsets of $X(\omega)$ of the form $\gamma_n^+(x)$, $n \in \mathbb{Z}$, $x \in X$; similarly, $\mathfrak{C}^-_{n,\omega}$ is the collection of all subsets of the form $\gamma_n^-(x)$. Set

$$\mathfrak{C}^+_{\omega} = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}^+_{n,\omega}; \mathfrak{C}^-_{\omega} = \bigcup_{n\in\mathbb{Z}} \mathfrak{C}^-_{n,\omega}. \tag{58}$$

Just as in the periodic case, the collections $\mathfrak{C}^+_{n,\omega}$, $\mathfrak{C}^-_{n,\omega}$, $\mathfrak{C}^+_{\omega}$, $\mathfrak{C}^-_{\omega}$ are semi-rings.

Remark. To make notation lighter, we shall often omit the subscript ω and only include it when dependence on ω is underlined.

3.1 Measures and Cocycles.

Let σ be the shift on Ω given by the formula $(\sigma\omega)_n = \omega_{n+1}$. Let $\mathbb{P}$ be an ergodic σ-invariant probability measure on Ω. We then have a natural cocycle $\mathbb{A}$ on the system $(\Omega, \sigma, \mathbb{P})$ defined, for $n > 0$, by the formula

$$\mathbb{A}(n, \omega) = A(\omega_n) \dots A(\omega_1).$$

The cocycle $\mathbb{A}$ will be called the *renormalization cocycle.*

We need the following assumptions on the measure $\mathbb{P}$ and on the cocyle $\mathbb{A}$.

Assumption 1 *The matrices $A(\omega_n)$ are almost surely invertible with respect to $\mathbb{P}$. There exists $\Gamma \in \mathfrak{G}$ such that $\mathbb{P}(\Gamma) > 0$.*

Assumption 2 *The logarithm of the renormalization cocycle (and of its inverse) is integrable.*

For $n < 0$ set

$$\mathbb{A}(n, \omega) = A^{-1}(\omega_{-n}) \dots A^{-1}(\omega_0).$$

and set $\mathbb{A}(0, \omega)$ to be the identity matrix.

The *transpose* cocycle $\mathbb{A}^t$ over the dynamical system $(\Omega, \sigma^{-1}, \mathbb{P})$ defined, for $n > 0$, by the formula

$$\mathbb{A}^t(n, \omega) = A^t(\omega_{1-n}) \dots A^t(\omega_0).$$

Similarly, for $n < 0$ write

$$\mathbb{A}^t(n, \omega) = (A^t)^{-1}(\omega_{-n}) \dots (A^t)^{-1}(\omega_1).$$

and set $\mathbb{A}^t(0, \omega)$ to be the identity matrix.

By Assumptions 1, 2, for $\mathbb{P}$-almost any $\omega \in \Omega$ we have the decompositions

$$\mathbb{R}^m = E^+_\omega \oplus E^-_\omega;\ \mathbb{R}^m = \tilde{E}^+_\omega \oplus \tilde{E}^-_\omega,$$

where E^+ is the Lyapunov subspace corresponding to positive Lyapunov exponents of $\mathbb{A}$; $\tilde{E}^+$ is the Lyapunov subspace corresponding to positive Lyapunov exponents of $\mathbb{A}^t$; E^- is the Lyapunov subspace corresponding to zero and negative Lyapunov exponents of $\mathbb{A}$; E^- is the Lyapunov subspace corresponding to

zero and negative Lyapunov exponents of $\mathbb{A}^t$. The standard inner product on $\mathbb{R}^m$ yields a nondegenerate pairing between the spaces E_ω^+ and $\tilde{E}_\omega^+$.

In particular, by Assumption 1, the spaces E_ω^+ and $\tilde{E}_\omega^+$ each contain a unique vector all whose coordinates are positive; we denote these vectors by $h^{(\omega)}$ and $\lambda^{(\omega)}$, respectively, and assume that they are normalized by (12).

Let $v \in E_\omega^+$ and for all $n \in \mathbb{Z}$ set $v^{(n)} = \mathbb{A}(n, \omega)v$. Introduce a finitely-additive complex-valued measure Φ_v^+ on the semi-ring $\mathfrak{C}_\omega^+$ (defined in (58)) by the formula

$$\Phi_v^+(\gamma_{n+1}^+(x)) = (v^{(n)})_{F(x_{n+1})}. \tag{59}$$

As before, the measure Φ_v^+ is invariant under holonomy along $\mathcal{F}^-$: by definition, we have the following

Proposition 19 *If $F(x_n) = F(x'_n)$, then $\Phi_v^+(\gamma_n^+(x)) = \Phi_v^+(\gamma_n^+(x'))$.*

The measures Φ_v^+ span a complex linear space, which is denoted $\mathcal{Y}_\omega^+$. The map $\mathcal{I}_\omega : v \to \Phi_v^+$ is an isomorphism between E_ω^+ and $\mathcal{Y}_\omega^+$. Set $\Phi_{1,\omega}^+ = \mathcal{I}_\omega(h^{(\omega)})$.

Now for $\tilde{v} \in \tilde{E}^+$ and for all $n \in \mathbb{Z}$ set $\tilde{v}^{(n)} = \mathbb{A}^t(n, \omega)\tilde{v}$ and introduce a finitely-additive complex-valued measure $\Phi_{\tilde{v}}^-$ on the semi-ring $\mathfrak{C}_\omega^-$ (defined in (58)) by the formula

$$\Phi_{\tilde{v}}^-(\gamma_n^-(x)) = (\tilde{v}^{(-n)})_{I(x_n)}. \tag{60}$$

By definition, the measure $\Phi_{\tilde{v}}^-$ is invariant under holonomy along $\mathcal{F}^+$: more precisely, we have the following

Proposition 20 *If $I(x_n) = I(x'_n)$, then $\Phi_{\tilde{v}}^-(\gamma_n^-(x)) = \Phi_{\tilde{v}}^-(\gamma_n^-(x'))$.*

Let $\mathcal{Y}_\omega^-$ be the space spanned by the measures $\Phi_{\tilde{v}}^-$, $\tilde{v} \in \tilde{E}^+$. The map $\tilde{\mathcal{I}}_\omega : \tilde{v} \to \Phi_{\tilde{v}}^-$ is an isomorphism between $\tilde{E}_\omega^+$ and $\mathcal{Y}_\omega^-$. Set $\Phi_{1,\omega}^- = \tilde{\mathcal{I}}_\omega(\lambda^{(\omega)})$.

Define a map $t_\sigma : X_\omega \to X_{\sigma\omega}$ by $(t_\sigma x)_i = x_{i+1}$. The map t_σ induces a map $t_\sigma^* : \mathcal{Y}_{\sigma\omega}^+ \to \mathcal{Y}_\omega^+$ given, for $\Phi_{\sigma\omega}^+ \in \mathcal{Y}_{\sigma\omega}^+$ and $\gamma \in \mathfrak{C}_\omega^+$, by the formula

$$t_\sigma^* \Phi^+(\gamma) = \Phi_{\sigma\omega}^+(t_\sigma \gamma).$$

We have the following commutative diagrams:

$$\begin{tikzcd} E_\omega^+ \arrow[r, "\mathcal{I}_\omega"] \arrow[d, "\mathbb{A}(1,\omega)"] & \mathcal{Y}_\omega^+ \\ E_{\sigma\omega}^+ \arrow[r, "\mathcal{I}_{\sigma\omega}"] & \mathcal{Y}_{\sigma\omega}^+ \arrow[u, "t_\sigma^*"'] \end{tikzcd}$$

$$\begin{tikzcd} \tilde{E}_\omega^+ \arrow[r, "\tilde{\mathcal{I}}_\omega"] & \mathcal{Y}_\omega^- \\ \tilde{E}_{\sigma\omega}^+ \arrow[r, "\tilde{\mathcal{I}}_{\sigma\omega}"] \arrow[u, "\mathbb{A}^t(1,\sigma\omega)"] & \mathcal{Y}_{\sigma\omega}^- \arrow[u, "t_\sigma^*"'] \end{tikzcd}$$

3.2 Pairings and weakly Lipschitz functions.

Given $\Phi^+ \in \mathcal{Y}^+_\omega$, $\Phi^- \in \mathcal{Y}^-_\omega$, introduce a finitely additive measure $\Phi^+ \times \Phi^-$ on the semi-ring $\mathfrak{C}$ of cylinders in $X(\omega)$: for any $C \in \mathfrak{C}$ and $x \in C$, set

$$\Phi^+ \times \Phi^-(C) = \Phi^+(\gamma^+_\infty(x) \cap C) \cdot \Phi^-(\gamma^-_\infty(x) \cap C). \tag{61}$$

Note that by Propositions 19, 20, the right-hand side in (61) does not depend on $x \in C$.

As above, for $\Phi^- \in \mathcal{Y}^-_\omega$, denote

$$m_{\Phi^-} = \Phi^+_1 \times \Phi^-. \tag{62}$$

In particular, we have a positive countably additive measure

$$\nu_\omega = \Phi^+_{h^{(\omega)}} \times \Phi^-_{\lambda^{(\omega)}}.$$

There is a natural $\mathbb{C}$-linear pairing $<,>$ between the spaces $\mathcal{Y}^+_\omega$ and $\mathcal{Y}^-_\omega$: for $\Phi^+ \in \mathcal{Y}^+_\omega$, $\Phi^- \in \mathcal{Y}^-_\omega$, set

$$< \Phi^+, \Phi^- >= \Phi^+ \times \Phi^-(X(\omega)). \tag{63}$$

As in Sec. 2.3, we have

Proposition 21 *Let $v \in E^+_\omega$, $\tilde{v} \in \tilde{E}^+_\omega$, $\Phi^+_v = \mathcal{I}_\omega(v)$, $\Phi^-_{\tilde{v}} = \tilde{\mathcal{I}}_\omega(\tilde{v})$. Then*

$$< \Phi^+_v, \Phi^-_{\tilde{v}} >= \sum_{i=1}^m v_i \tilde{v}_i. \tag{64}$$

The pairing $<,>$ is non-degenerate and t^*_σ-invariant.

The function space $Lip^+_w(X(\omega))$ is introduced in the same way as before: a bounded Borel-measurable function $f : X(\omega) \to \mathbb{C}$ belongs to the space $Lip^+_w(X)$ if there exists a constant $C > 0$ such that for all $n \geq 0$ and any $x, x' \in X$ satisfying $F(x_{n+1}) = F(x'_{n+1})$, we have

$$\left| \int_{\gamma^+_n(x)} f d\Phi^+_1 - \int_{\gamma^+_n(x')} f d\Phi^+_1 \right| \leq C, \tag{65}$$

and, if C_f is the infimum of all C satisfying (65), then we norm $Lip^+_w(X)$ by setting

$$||f||_{Lip^+_w} = \sup_X f + C_f.$$

As before, we denote by $Lip^+_{w,0}(X(\omega))$ the subspace of functions of ν_ω-integral zero.

Take $\Phi^- \in \mathcal{Y}^-$. Any function $f \in Lip^+_w(X)$ is integrable with respect to the measure m_{Φ^-} in the same sense as in Sec. 2.4, and a measure $\Phi^+_f \in \mathcal{Y}^+$ is defined by the requirement that for any $\Phi^- \in \mathcal{Y}^-$ we have

$$< \Phi^+_f, \Phi^- >= \int_{X(\omega)} f dm_{\Phi^-}. \tag{66}$$

Note that the mapping $\Xi^+_\omega : Lip^+_w(X(\omega)) \to \mathcal{Y}^+_\omega$ given by $\Xi^+_\omega(f) = \Phi^+_f$ is continuous by definition and satisfies

$$\Xi^+_{\sigma\omega}(f \circ t_\sigma) = (t_\sigma)^* \Xi^+_\omega(f). \tag{67}$$

From the definitions we also have

Proposition 22 *Let $\Phi^+(1), \dots, \Phi^+(r)$ be a basis in $\mathcal{Y}^+_\omega$ and let $\Phi^-(1), \dots, \Phi^-(r)$ be the dual basis in $\mathcal{Y}^-_\omega$ with respect to the pairing $<,>$. Then for any $f \in Lip^+_w(X(\omega))$ we have*

$$\Phi^+_f = \sum_{i=1}^{r} \big(m_{\Phi^-(i)}(f)\big)\Phi^+(i).$$

3.3 Orderings and flows.

Assume that for $\mathbb{P}$-almost every ω a partial ordering $\mathfrak{o}(\omega)$ is given on $\mathcal{E}(\omega_n)$ for all $n \in \mathbb{Z}$ in such a way that edges starting at a given vertex are ordered linearly, while edges starting at different vertices are incomparable. Assume, moreover, that the orders $\mathfrak{o}(\omega)$ are σ-invariant, in the sense that the ordering $\mathfrak{o}(\omega)$ on $\mathcal{E}(\omega_n)$ is the same as the ordering $\mathfrak{o}(\sigma\omega)$ on $\mathcal{E}((\sigma\omega)_{n-1})$.

Similarly to the above, construct spaces $X_\mathfrak{o}(\omega)$ and introduce a flow $h_t^{(+,\omega)}$ on each $X_\mathfrak{o}(\omega)$. The shift σ renormalizes the flows $h_t^{(+,\omega)}$: if we set

$$H^{(1)}(n,\omega) = ||\mathbb{A}(n,\omega)||, \tag{68}$$

then for any $t \in \mathbb{R}$ we have a commutative diagram

$$\begin{array}{ccc} X(\omega) & \xrightarrow{h_t^{(+,\omega)}} & X(\omega) \\ \Big\downarrow t_\sigma & & \Big\downarrow t_\sigma \\ X(\sigma\omega) & \xrightarrow{h^{(+,\sigma\omega)}_{t/H^{(1)}(1,\omega)}} & X(\sigma\omega) \end{array}$$

As before, each measure $\Phi^+ \in \mathcal{Y}^+_\omega$ yields a Hölder cocycle over the flow $h_t^{(+,\omega)}$; we shall denote the cocycle by the same letter as the measure.

Note that for any $\Phi^- \in \mathcal{Y}^-_\omega$ the measure m_{Φ^-} defined by (62) satisfies

$$(h_t^{(+,\omega)})_* m_{\Phi^-} = m_{\Phi^-},$$

similarly to G. Forni's invariant distributions [5], [6].

Note that the mapping $\Xi^+_\omega : Lip^+_w(X(\omega)) \to \mathcal{Y}^+_\omega$ given by $\Xi^+_\omega(f) = \Phi^+_f$ by definition satisfies

$$\Xi^+_\omega(f \circ h_t^{(+,\omega)}) = \Xi^+_\omega(f). \tag{69}$$

We thus have the following

Theorem 7 *Let $\mathbb{P}$ be an ergodic σ-invariant probability measure on Ω satisfying the assumptions 1, 2. For any $\varepsilon > 0$ there exists a positive constant C_ε depending only on $\mathbb{P}$ such that the following holds. For $\mathbb{P}$-almost any ω there exists a continuous mapping $\Xi^+_\omega : Lip^+_w(X(\omega)) \to \mathcal{Y}^+_\omega$ such that for any $f \in Lip^+_w(X(\omega))$, any $x \in X(\omega)$ and all $T > 0$ we have*

$$|\int_0^T f \circ h_t^{(+,\omega)}(x)dt - \Xi^+_\omega(f)\big(x,t\big)| \leq C_\varepsilon ||f||_{Lip^+_w}(1 + T^\varepsilon).$$

The mapping Ξ^+_ω satisfies the equality $\Xi^+_\omega(f \circ h_t^{(+,\omega)}) = \Xi^+_\omega(f)$. The diagram

$$\begin{array}{ccc} Lip^+_w(X(\sigma\omega)) & \xrightarrow{\Xi^+_{\sigma\omega}} & \mathcal{Y}^+_{\sigma\omega} \\ \Big\downarrow t^*_\sigma & & \Big\downarrow t^*_\sigma \\ Lip^+_w(X(\omega)) & \xrightarrow{\Xi^+_\omega} & \mathcal{Y}^+_\omega \end{array}$$

is commutative.

The mapping Ξ^+_ω is given by $\Xi^+_\omega(f) = \Phi^+_f$, where Φ^+_f is defined by (66).

Now assume that the second Lyapunov exponent θ_2 of the renormalization cocycle $\mathbb{A}$ is positive and simple. Let $v_2 \in E^+_\omega$ be a Lyapunov vector corresponding to the exponent $\exp(\theta_2)$ (such a vector is defined up to multiplication by a scalar). Introduce a multiplicative cocycle $H^{(2)}(n,\omega)$ over σ by the formula

$$H^{(2)}(n,\omega) = \frac{|\mathbb{A}(n,\omega)v_2^{(\omega)}|}{|v_2^{(\omega)}|}. \tag{70}$$

Recall that the cocycle $H^{(1)}(n,\omega)$ is given by (68). Similarly to the above, given a bounded measurable function $f : X(\omega) \to \mathbb{R}$ and $x \in X(\omega)$, introduce a continuous function $\mathfrak{S}_n[f,x]$ on the unit interval by the formula

$$\mathfrak{S}_n[f,x](\tau) = \int\limits_0^{\tau H^{(1)}(n,\omega)} f \circ h_t^{(+,\omega)}(x)dt. \tag{71}$$

The functions $\mathfrak{S}_n[f,x]$ are $C[0,1]$-valued random variables on the probability space $(X(\omega),\nu_\omega)$.

Theorem 8 *Let $\mathbb{P}$ be an ergodic σ-invariant probability measure on Ω satisfying the assumptions 1, 2 and such the second Lyapunov exponent of the renormalization cocycle $\mathbb{A}$ with respect to $\mathbb{P}$ is positive and simple.*

For $\mathbb{P}$-almost any $\omega' \in \Omega$ there exists a non-degenerate compactly supported measure $\eta_{\omega'}$ on $C[0,1]$ and, for $\mathbb{P}$-almost any pair (ω,ω') there exists a sequence of moments $l_n = l_n(\omega,\omega')$ such that the following holds.

For $\mathbb{P}$-almost any ω there exists a continuous functional

$$\mathfrak{a}^{(\omega)} : Lip^+_w(X(\omega)) \to \mathbb{R}$$

such that for $\mathbb{P}$-almost any ω' and any $f \in Lip^+_{w,0}(X(\omega))$ satisfying $\mathfrak{a}^{(\omega)}(f) \neq 0$ the sequence of random variables

$$\frac{\mathfrak{S}_{l_n(\omega,\omega')}[f,x]}{\mathfrak{a}^{(\omega)}(f)H^{(2)}(l_n(\omega,\omega'),\omega)}$$

converges in distribution to $\eta_{\omega'}$ as $n \to \infty$.

Theorems 7, 8 imply Theorems 3, 4. The proofs of Theorems 7, 8 follow the same pattern as those of Theorems 5, 6; detailed proofs will appear in the sequel to this paper.

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